Metrical Quantization[*]

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Abstract

Canonical quantization may be approached from several different starting points. The usual approaches involve promotion of c-numbers to q-numbers, or path integral constructs, each of which generally succeeds only in Cartesian coordinates. All quantization schemes that lead to Hilbert space vectors and Weyl operators—even those that eschew Cartesian coordinates—implicitly contain a metric on a flat phase space. This feature is demonstrated by studying the classical and quantum "aggregations", namely, the set of all facts and properties resident in all classical and quantum theories, respectively. Metrical quantization is an approach that elevates the flat phase space metric inherent in any canonical quantization to the level of a postulate. Far from being an unwanted structure, the flat phase space metric carries essential physical information. It is shown how the metric, when employed within a continuous-time regularization scheme, gives rise to an unambiguous quantization procedure that automatically leads to a canonical coherent state representation. Although attention in this paper is confined to canonical quantization we note that alternative, nonflat metrics may also be used, and they generally give rise to qualitatively different, noncanonical quantization schemes.

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1 Introduction

Quantization, like any other procedure, lends itself to an axiomatization. As discussed shortly, there are such procedures that characterize the usual quantization proposals of Heisenberg, Schrödinger, and Feynman. Hidden in these procedures is an often unstated assumption, namely, that the coordinates in which the very quantization rules are laid down must be chosen to be Cartesian whenever a canonical quantization is sought. This procedural step is so ingrained and automatic that it is often overlooked or ignored for what it really is, namely, an essential assumption in the given procedure. In this paper we briefly review postulates of the usual quantization procedures and introduce yet another procedure we refer to as metrical quantization.

Let us start with a brief review of classical mechanics.

1.1 Classical mechanics

Consider a phase space $\mathcal{M}$ for a single degree of freedom which is two dimensional. As a symplectic manifold the space $\mathcal{M}$ is endowed with a symplectic two form ω, which is nondegenerate and closed, $d\omega = 0$. Darboux's Theorem assures us that local coordinates p and q exist such that $\omega = dp \wedge dq$ in the given coordinates. Such coordinates are referred to as canonical coordinates, and any coordinate transformation with a unit Jacobian leads from one set of canonical coordinates to another set of canonical coordinates. Indeed, if r and s denote another pair of canonical coordinates, then it follows that $rds = pdq + dF(s, q)$ for some generator F. The new coordinates are canonical since the exterior derivative of both sides of this relation yields $dr \wedge ds = dp \wedge dq = \omega$.

Besides the kinematical aspects of the classical theory of mechanics, dynamics arises with the introduction of a distinguished scalar, the Hamiltonian H, or as expressed in the original canonical coordinates, the function $H(p, q)$. By a scalar we mean that $\overline{H}(r, s) \equiv H(p(r, s), q(r, s)) = H(p, q)$, an equation which indicates how H transforms under (canonical) coordinate transformations. Finally, classical dynamics may be introduced as the stationary paths of a distinguished action functional given in coordinate form by

$$I = \int [p\dot{q} + \dot{G}(p, q) - H(p, q)]\, dt \, , \qquad (1)$$

subject to variations that hold both $p(t)$ and $q(t)$ fixed at the initial time

$t = 0$ and the final time $t = T$. The resultant equations are independent of the gauge function G, and are given by

$$\dot{q} = \partial H(p,q)/\partial p \, ,$$
$$\dot{p} = -\partial H(p,q)/\partial q \, . \tag{2}$$

Note that the exterior derivative of the one form $pdq + dG(p,q)$ that appears in the action functional leads to $d[pdq + dG(p,q)] = dp \wedge dq = \omega$. In this way the symplectic structure enters the dynamics.

Lastly, we observe that the dynamical equations of motion may also be given a Poisson bracket structure. In particular, if

$$\{A, B\} \equiv \frac{\partial A}{\partial q} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial q} \, , \tag{3}$$

then it follows that

$$\dot{q} = \{q, H(p,q)\} \, ,$$
$$\dot{p} = \{p, H(p,q)\} \, , \tag{4}$$

and for a general function $W(p,q)$ it follows that

$$\dot{W}(p,q) = \{W(p,q), H(p,q)\} \, . \tag{5}$$

Thus, since $\{B, A\} = -\{A, B\}$, we observe that

$$\dot{H}(p,q) = \{H(p,q), H(p,q)\} = 0 \, , \tag{6}$$

and therefore $H(p,q) = E$, which is a constant of the motion usually identified with the energy.

2 The Classical Mechanics Aggregation

Let us collect all the concepts and formulas appropriate to classical mechanics, a few of which have been indicated above, in one place, and let us refer to that as the *classical mechanics aggregation*. For example, the classical mechanics aggregation would include the set of all canonical coordinates, the set of all Hamiltonians each of which is expressed in all possible canonical

coordinates, the rules for dynamical evolution, and indeed the set of all solutions of the dynamical equations of motion for each Hamiltonian expressed in all possible canonical coordinates. Also in the classical mechanics aggregation would be the formulation of classical mechanics expressed in differential geometric form, i.e., as coordinate-free expressions and operations that effect the Poisson brackets, etc. Evidently, the classical mechanics aggregation contains all that is known and, implicitly, all that is knowable about classical mechanics!

Let us develop an analogous aggregation appropriate to quantum mechanics.

3 The Quantum Mechanics Aggregation

There are a number of standard ideas and equations that enter into the formulation of quantum mechanics irrespective of the particular details of the system being quantized, and, for purposes of illustration, let us focus on systems with just one degree of freedom. We have in mind, for example, a Hilbert space composed of complex, square integrable functions over the real line, namely the space $L^2(\mathbb{R})$, or a Hilbert space composed of square summable sequences, namely the space l^2, etc. Operators arise in the form of functions of position and derivatives with respect to position, or functions of momentum and derivatives with respect to momentum, or semi-infinite square matrices, etc. Probability amplitudes occur in the form of inner products of two Hilbert space vectors, or more generally, matrix elements of an operator in the form of an inner product involving two vectors with an operator standing between them. Many of these concepts can be formulated in a coordinate-free language in terms of an abstract Hilbert space formulation and an abstract operator language as well. These elements form the arena in which quantum mechanics takes place. Quantum mechanics is also distinguished by equivalent sets of rules for the introduction of dynamics. For example, there is the abstract Schrödinger equation giving the time derivative of the state vector as the action of the Hamiltonian operator on the state vector, apart from suitable constants $(i\hbar)$. Alternatively, there is the Heisenberg equation of motion which equates the time derivative of an operator in the Heisenberg picture to the commutator of the operator with the Hamiltonian, again up to the same constants. Additionally, we mention the Feynman

representation of the propagator as a path integral, a representation which in fact is a direct consequence of the abstract vector and operator language, or alternatively, a consequence of the Schrödinger equation and its solution for a suitable boundary condition.

We may also mention distinguished operator sets such as the Heisenberg canonical operators P and Q which, either abstractly or in a concrete realization, satisfy the fundamental commutation relation $[Q, P] = i\hbar\mathbb{1}$. If these operators are self adjoint then we may also consider the Weyl operators $U[p, q] \equiv \exp[i(pQ - qP)/\hbar]$ for all real p and q. Armed with such operators and an arbitrary normalized vector in the Hilbert space $|\eta\rangle$, we may consider the canonical coherent states

$$|p, q\rangle \equiv |p, q; \eta\rangle \equiv e^{i(pQ-qP)/\hbar}|\eta\rangle . \tag{7}$$

It is but a simple exercise to show that

$$\int |p, q\rangle\langle p, q|\, dpdq/2\pi\hbar = \mathbb{1} \tag{8}$$

for any choice of the fiducial vector $|\eta\rangle$. Thus, coherent states, the representations of Hilbert space they induce, etc., are all implicitly contained within the *quantum mechanical aggregation*. Unitary transformations that map one form of Hilbert space vectors and one form of operators into another form are all part of the quantum mechanical aggregation. In short, everything kinematical and dynamical that one could think of belonging to Hilbert space, operator theory, quantum mechanics, etc., everything known and, implicitly, everything knowable about quantum mechanics is contained in the quantum mechanical aggregation.

Now let us try to build a bridge between the classical mechanical aggregation and the quantum mechanical aggregation.

4   Conventional Quantization

The act of quantization is designed to connect the principal entities in the classical mechanical aggregation with the appropriate entities in the quantum mechanical aggregation, in some cases in a one-to-one fashion, but in other cases in a many-one fashion. It is the genius of Heisenberg and Schrödinger that they were able to guess several basic concepts and quantities lying in the

quantum mechanical aggregation and use these few ideas as stepping stones in order to construct a bridge between the classical and the quantum worlds. Feynman used a different set of concepts and quantities to select his stepping stones between these two worlds. In modern parlance, we could call these stepping stones "postulates" (or at the very least "assumptions").

Heisenberg quantization

In the case of Heisenberg quantization, we may cast the postulates in the form (for postulate 1. see below):

2. Introduce matrices $Q = \{Q_{mn}\}$ and $P = \{P_{mn}\}$, where $m, n \in \{1, 2, 3, \ldots\}$, that satisfy $[Q, P]_{mn} \equiv \Sigma_p(Q_{mp}P_{pn} - P_{mp}Q_{pn}) = i\hbar\delta_{mn}$.

3. Build a Hamiltonian matrix $H = \{H_{mn}\}$ as a function (e.g., polynomial) of the matrices, $H_{mn} = H(P, Q)_{mn}$, that is the same function as the classical Hamiltonian $H(p, q)$. (In so doing there may be operator ordering ambiguities which this prescription cannot resolve; choose an ordering that leads to a Hermitian operator.)

4. Introduce the equation of motion $i\hbar\dot{X}_{mn} = [X, H]_{mn}$ for the elements of a general matrix $X = \{X_{mn}\}$. □

Along with these postulates comes the implicit task of solving the called for equations of motion subject to suitable operator-valued boundary conditions. Once the several steps are accomplished, a general path has opened up as to how a given system is to be taken from its classical version to its quantum version.

Accepting these postulates, it becomes clear how the general classical system is to be connected with the general quantum system *apart from one postulate that we have neglected and which was not immediately obvious to the founding fathers.* The question arises as to *exactly which choice of canonical coordinates are to be used when promoting the classical canonical variables to quantum canonical variables.* After the principal paper on quantization [1], it subsequently became clear to Heisenberg that *it is necessary to make this promotion from c-number to q-number variables only in Cartesian coordinates.* Thus there is implicitly another postulate [2]:

1. Express the classical kinematical variables p and q in Cartesian coordinates prior to promoting them to matrices $\{P_{mn}\}$ and $\{Q_{mn}\}$, respectively.

We will present a rationale for this postulate below.

Schrödinger quantization

The postulates for Schrödinger's formulation of quantization may be given in the following form [3]:

1. Express the classical kinematical variables p and q in Cartesian coordinates.

2. Promote the classical momentum p to the differential operator $-i\hbar(\partial/\partial x)$ and the classical coordinate q to the multiplication operator x, a choice that evidently satisfies the commutation relation $[x, -i\hbar(\partial/\partial x)] = i\hbar$.

3. Define the Hamiltonian operator $\mathcal{H}$ as the classical Hamiltonian with the momentum variable p replaced by the operator $-i\hbar(\partial/\partial x)$ and the coordinate variable q replaced by the operator x. (In so doing there may be operator ordering ambiguities which this prescription cannot resolve; choose an ordering that leads to a Hermitian operator.)

4. For $\psi(x)$ a complex, square integrable functions of x, introduce the dynamical equation $i\hbar\dot{\psi} = \mathcal{H}\psi$. $\square$

Implicit with these postulates is the instruction to solve the Schrödinger equation for a dense set of initial conditions and a large class of Hamiltonian operators, and in that way help to build up the essentials of the quantum mechanical aggregation.

It is interesting to note that Schrödinger himself soon became aware of the fact that his procedure generally works only in Cartesian coordinates.

Feynman quantization

Feynman's formulation of quantization focuses on the solution to the Schrödinger equation and postulates that the propagator, an integral kernel that maps the wave function (generally in the Schrödinger representation) at one time to the wave function at a later time, may be given by means of a path integral expression [4]. On the surface, it would seem that the (phase space) path integral, using only concepts from classical mechanics, would seem to get around the need for Cartesian coordinates; as we shall see that is not the case. As postulates for a path integral quantization scheme we have:

1. Express the classical kinematical variables p and q in Cartesian coordinates.

2. Given that $|q, t\rangle$, where $Q(t)|q, t\rangle = q|q, t\rangle$, denote sharp position eigenstates, write the transition matrix element in the form of a path integral as

$$\langle q'', T|q', 0\rangle = \mathcal{M} \int \exp\{(i/\hbar)\int[p\dot{q} - H(p, q)]\, dt\}\, \mathcal{D}p\, \mathcal{D}q\, . \qquad (9)$$

3. Recognize that the formal path integral of Step 2. is *effectively undefined* and replace it by a *regularized* form of path integral, namely,

$$\langle q'', T|q', 0\rangle = \lim_{N \to \infty} M_N \int \exp\{(i/\hbar)\Sigma_{l=0}^{N}[p_{l+1/2}(q_{l+1} - q_l)$$
$$-\epsilon H(p_{l+1/2}, (q_{l+1} + q_l)/2)]\}\, \Pi_{l=0}^{N} dp_{l+1/2}\, \Pi_{l=1}^{N} dq_l\, , \qquad (10)$$

where $q_{N+1} = q''$, $q_0 = q'$, $M_N = (2\pi\hbar)^{-(N+1)}$, and $\epsilon = T/(N + 1)$. $\square$

Implicit in the latter expression is a Weyl ordering choice to resolve any operator ordering ambiguities. Observe that the naive lattice formulation of the classical action leads to correct quantum mechanical results, generally speaking, only in Cartesian coordinates. Although the formal phase space path integral of postulate 2. appears superficially to be covariant under canonical coordinate transformations, it would be incorrect to conclude that was the case inasmuch as it would imply that the spectrum of diverse physical systems would be identical. In contrast, the naive lattice prescription applies only to Cartesian coordinates, the same family of coordinates singled out in the first postulate of each of the previous quantization schemes.

4.1   Elements of the quantum mechanical aggregation

Traditional quantization—be it Heisenberg, Schrödinger, or Feynman—leads invariably to a Hilbert space (or a particular representation thereof), and to canonical operators (or particular representations thereof). For physical reasons we restrict attention to that subclass of systems wherein the canonical operators are self adjoint and obey not only the Heisenberg commutation relations but also the more stringent Weyl form of the commutation relations. In particular, we assert that a byproduct of any conventional quantization scheme—and even some nonconventional quantization schemes such as

geometric quantization or deformation quantization—is to lead to (normalized) Hilbert space vectors, say $|\eta\rangle$, and a family of unitary Weyl operators $U[p,q] \equiv \exp[i(pQ - qP)/\hbar]$, $(p,q) \in \mathbb{R}^2$, that obey the standard Weyl commutation relation. These expressions lead directly to a set of coherent states each of the form $|p,q\rangle \equiv U[p,q]|\eta\rangle$. Given such conventional quantities lying in the quantum mechanical aggregation, and minimal domain assumptions, we first build the one form

$$\theta(p,q) \equiv i\hbar\langle p,q|d|p,q\rangle = \tfrac{1}{2}(p\,dq - q\,dp) + \langle P\rangle\,dq - \langle Q\rangle\,dp\,, \qquad (11)$$

where $\langle(\,\cdot\,)\rangle \equiv \langle\eta|(\,\cdot\,)|\eta\rangle$, and which is recognized as a natural candidate for the classical symplectic potential for a general $|\eta\rangle$. Indeed, $d\theta = dp \wedge dq = \omega$ holds for a general $|\eta\rangle$. As a second quantity of interest, we build the Fubini-Study metric

$$d\sigma^2(p,q) \equiv 2\hbar^2[\||d|p,q\rangle\|^2 - |\langle p,q|d|p,q\rangle|^2]$$
$$= 2\langle(\Delta Q)^2\rangle dp^2 + 2\langle(\Delta P)(\Delta Q) + (\Delta Q)(\Delta P)\rangle dpdq + 2\langle(\Delta P)^2\rangle dq^2. (12)$$

Here $\Delta Q \equiv Q - \langle Q\rangle$, etc. The latter expression given above holds for a general vector $|\eta\rangle$. Observe well, for a general $|\eta\rangle$, that this phase space metric is always *flat* because all the metric coefficients are constants. Stated otherwise, for a general $|\eta\rangle$, the Fubini-Study metric invariably describes a *flat phase space*, here expressed in (almost) Cartesian coordinates thanks to the use of canonical group coordinates for the Weyl group. For "physical" vectors $|\eta\rangle$, defined such that $\langle(\Delta Q)^2 + (\Delta P)^2\rangle = \mathrm{o}(\hbar^0)$, it follows that the phase space metric is a *quantum property* and it vanishes in the limit $\hbar \to 0$; indeed, if $|\eta\rangle$ is chosen as the ground state of a harmonic oscillator with unit angular frequency, then $d\sigma^2 = \hbar(dp^2 + dq^2)$. One may of course change the coordinates, e.g., introduce $r = r(p,q)$ and $s = s(p,q)$; this may change the form of the metric coefficients for $d\sigma^2$, but it will not alter the fact that the underlying phase space is still flat.

We conclude these remarks by emphasizing that inherent in any canonical quantization scheme is the implicit assumption of a flat phase space which can carry globally defined Cartesian coordinates. These properties automatically lie within the quantum mechanical aggregation for any quantization scheme that leads to Hilbert space vectors and canonical operators!

5 Metrical Quantization

We define metrical quantization by the following set of postulates:

   1. Assign to classical phase space a flat space metric $d\sigma^2$, and choose Cartesian coordinates in such a way that

$$d\sigma^2(p,q) = \hbar(dp^2 + dq^2) . \tag{13}$$

   2. Introduce the regularized phase-space path integral, which explicitly uses the phase space metric, and is formally given by

$$K(p'',q'',T;p',q',0) = \lim_{\nu\to\infty} \mathcal{N}_\nu \int \exp\{(i/\hbar)\textstyle\int[(p\dot{q}-q\dot{p})/2 - h(p,q)]\,dt\}$$
$$\times \exp\{-(1/2\nu)\textstyle\int[\dot{p}^2+\dot{q}^2]\,dt\}\,\mathcal{D}p\,\mathcal{D}q \tag{14}$$

and more precisely given by

$$K(p'',q'',T;p'q',0)$$
$$= \lim_{\nu\to\infty} 2\pi\hbar e^{\nu T/2} \int \exp\{(i/\hbar)\textstyle\int[(pdq-qdp)/2 - h(p,q)\,dt]\}\,d\mu_W^\nu(p,q) , \tag{15}$$

where μ_W^ν denotes a Wiener measure for two-dimensional Brownian motion on the plane expressed in Cartesian coordinates, and where ν denotes the diffusion constant. Finally, we observe that a s a positive-definite function, it follows from the GNS (Gel'fand, Naimark, Segal) Theorem that

$$K(p'',q'',T;p',q',0) \equiv \langle p'',q''|\,e^{-i\mathcal{H}T/\hbar}\,|p',q'\rangle , \tag{16}$$
$$|p,q\rangle \equiv e^{i[(pQ-qP)/\hbar]}|0\rangle , \qquad [Q,P] = i\hbar\mathbb{1} , \tag{17}$$
$$(Q+iP)|0\rangle = 0 , \qquad \langle 0|0\rangle = 1 , \tag{18}$$
$$\mathcal{H} \equiv \textstyle\int h(p,q)\,|p,q\rangle\langle p,q|\,dp\,dq/2\pi\hbar . \tag{19}$$

All these things follow from positive-definiteness, and the implication is that the Wiener measure regularized phase-space path integral automatically gives rise to the propagator expressed in a coherent-state representation. □

   The canonical quantization formulation given above has raised the metric on a flat phase space to the level of a postulate. The assumption that the given coordinates are indeed Cartesian is by no means an arbitrary one. There is, in fact, a great deal of physics in the statement that certain coordinates are Cartesian. In the present case, we can read that physics straight

out of (19) which relates the classical Hamiltonian $h(p, q)$ to the quantum Hamiltonian operator $\mathcal{H}$. The given integral representation is in fact equivalent to antinormal ordering, i.e., the monomial $(q + ip)^k (q - ip)^l$ is quantized as the operator $(Q + iP)^k (Q - iP)^l$ for all nonnegative integers k and l. Thus, for example, in these coordinates, the c-number expression $p^2 + q^2 + q^4$ is quantized as $P^2 + Q^2 + Q^4 + \mathrm{O}(\hbar)$. Of course, the latter term can be made explicit; here we are only interested in the fact that the leading terms $[\mathrm{O}(\hbar^0)]$ of the quantum Hamiltonian operator are exactly those as given by the classical Hamiltonian. This connection may seem evident but that is far from the case.

Observe that expressed in terms of the Brownian motion regularization, and when we define the stochastic integral $\int p \, dq$ via a (midpoint) Stratonovich prescription (as we are free to do in Cartesian coordinates), the procedure of metrical quantization is actually *covariant* under canonical coordinate transformations. As noted earlier, such a transformation is determined by the expression $r \, ds = p \, dq + dF(s, q)$ in the classical theory, and, thanks to the S tratonovich prescription, also in the quantum theory where the paths p and q are Brownian motion paths. The function h transforms as a scalar, and therefore $\overline{h}(r, s) \equiv h(p(r, s), q(r, s)) = h(p, q)$. Lastly, we transform the Wiener measure which still describes Brownian motion on a flat two-dimensional plane, but now, generally speaking, in curvilinear coordinates. After the change of coordinates the propagator reads

$$\overline{K}(r'', s'', T; r', s', 0)$$
$$= \lim_{\nu \to \infty} 2\pi\hbar e^{\nu T/2} \int \exp\{(i/\hbar)\textstyle\int [r \, ds + dG(r, s) - \overline{h}(r, s) \, dt]\} \, d\overline{\mu}_W^\nu(r, s). \quad (20)$$

Here, dG denotes a total differential, which amounts to nothing more than a phase change of the coherent states, and $\overline{\mu}_W^\nu$ denotes Brownian measure on the flat two-dimensional plane expressed now in curvilinear coordinates rather than Cartesian coordinates. In this case the connection of the classical and quantum Hamiltonians is given by

$$\mathcal{H} = \textstyle\int \overline{h}(r, s) \, |p(r, s), q(r, s)\rangle\langle p(r, s), q(r, s)| \, dr \, ds / 2\pi\hbar \; . \quad (21)$$

Observe in this coordinate change that the coherent states have remained unchanged (only their names have changed) and, as a consequence, the Hamiltonian operator $\mathcal{H}$ is *absolutely unchanged* even though its c-number counterpart (symbol) is now expressed by $\overline{h}(r, s)$. In other words, the leading

[O($\hbar^0$)] dependence of $\overline{h}$ and $\mathcal{H}$ are no longer identical. As we have stressed elsewhere [5], the *physical significance* of the *mathematical expression* for a given classical quantity is *encoded* into the specific coordinate form of the auxiliary metric $d\sigma^2$; for example, if the metric is expressed in Cartesian coordinates, then the physical meaning of the classical Hamiltonian is that directly given by its coordinate form, as has been illustrated above by the anharmonic oscillator. Since quantization deals, for example, with the highly physical energy spectral values, it is *manditory* that the mathematical expression for the Hamiltonian somehow "know" to which physical system it belongs. It is the role of the metric and the very form of the metric coefficients themselves to keep track of just what physical quantity is represented by any given mathematical expression. And that very metric is build right into the Wiener measure regularized phase-space path integral, which, along with the metric itself, is the centerpiece of metrical quantization.

Although we do not develop the subject further here, it is noteworthy that choosing a different geometry to support the Brownian motion generally leads to a qualitatively different quantization. For example, if the two-dimensional phase space has the geometry of a sphere of an appropriate radius, then metrical quantization leads not to canonical operators but rather to spin (or angular momentum) kinematical operators that obey the Lie algebra commutation relations of SU(2). On the other hand, for a phase space with the geometry of a space of constant negative curvature, metrical quantization leads to kinematical operators that are the generators of the Lie algebra for SU(1,1). Stated otherwise, the geometry of the chosen metric in postulate 1. of metrical quantization—which then explicitly appears in the expression defining the Wiener measure regularization in postulate 2.—actually determines the very nature of the kinematical operators in the metrical quantization procedure [6].

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